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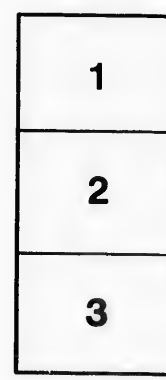
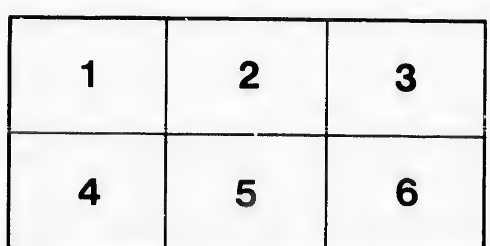




P. M. Lovelace.

Dec. 13/99.

Canadian Society of Civil Engineers.

INCORPORATED 1887.

ADVANCE PROOF—*(Subject to revision)*

TRANSITION CURVES.

BY

E. S. M. LOVELACE, B.A.Sc., A.M., Can. Soc. C. E

(To be read Thursday, December 7th, 1899.)

Some years ago the writer published a pamphlet on the "Lemniscata as a Transition Curve," a copy of which is in the possession of the Society. The pamphlet was rather favorably reviewed in the "Railroad Gazette," June 30th, 1893, and in "Engineering News," November 2nd of the same year, and as a result a good many copies were sold to engineers in different parts of the United States.

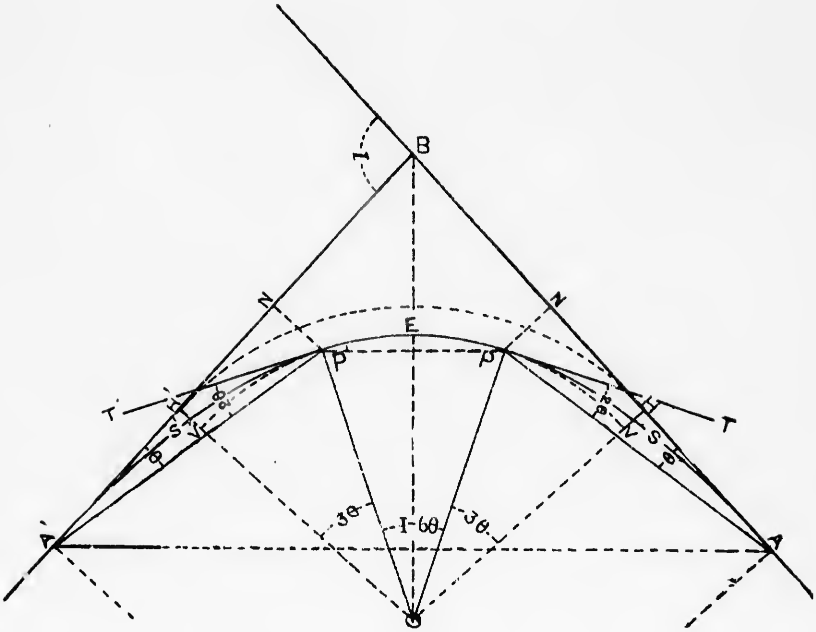
As was indicated, however, in the articles in the above-mentioned journals, and as the writer himself realized shortly after the book was printed, the tables given in it were not sufficiently extensive, while the general formulae giving the elements of the proposed curve were perhaps a little too complicated to lend themselves readily to work in the field, and consequently the pamphlet was not as successful as had been confidently hoped it would be. Before laying before the Society the modified set of formulae which the writer now proposes to use, it may be of interest to look at the simple way in which those given in the pamphlet might have been written, but which unfortunately was not discovered until too late.

Referring to the accompanying diagram and calling

the distance measured along chord A P	=	C
the distance measured along curve A S P	=	S
the offset distance H V	=	K
the radius of main curve O P	=	R
the deflection angle P A N	=	θ

the formulae given in pamphlet are as follows:—

$$(1)'' \quad \sin 2\theta = \frac{C}{3R}$$


B
I
N
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E
P
P
T'
T
H
S
V
2θ
θ
3θ
I-6θ
3θ
A'
A
O

$$(2)'' \; AH = \frac{c}{2} \left(\frac{2 + \cos 2\theta}{\cos \theta}\right)$$

$$(3)'' \; OH = \frac{c}{6} \left(\frac{2 - \cos 2\theta}{\sin \theta}\right)$$

$$(4)'' \; AB = OH \tan \frac{I}{2} + AH$$

$$(5)'' \; s = c + \frac{c^3}{90 \, R^2} + \frac{c^5}{1944 \, R^4}$$

In place of the above write:—

$$(1)' \; \sin 2\theta = \frac{c}{3 \, R}$$

$$(2)' \; AH = \frac{R}{2} \, (3 \sin \theta + \sin 3 \theta)$$

$$(3)' \; OH = \frac{R}{2} \, (3 \cos \theta - \cos 3 \theta)$$

$$(4)' \; AB = \frac{R}{2} \sec \frac{I}{2} \left[3 \sin \left(\frac{I}{2} + \theta\right) - \sin \left(\frac{I}{2} - 3\theta\right)\right]$$

$$(5)' \; s = \frac{c}{1 - \frac{2}{5} \theta^2} = c + \frac{20}{3} \frac{k^2}{c}$$

(6)′ deflection to point distant l along curve from $A = \frac{\theta l^2}{s^2}$

Equation (4) is the first expression giving the value of A B in one simple formula, which the writer has seen. It can be used to special advantage when the curves of an existing track are to be eased off in such a manner that the total length of the track remains the same, and, so, no rails will have to be cut. To insure this, looking at Fig. it will be necessary that:

$$R' \frac{I}{2} + \left[(AB - R' \tan \frac{I}{2}\right) = S + R \left(\frac{I}{2} - 3\theta\right)$$

where R' = radius of existing track

$$\text{that is } R' \left(\frac{I}{2} - \tan \frac{I}{2}\right) = s - AB + R \left(\frac{I}{2} - 3\theta\right)$$

substituting for S and A B, their values given by (5)′ and (6)′.

$$R = \frac{R^1 \left(\frac{I}{2} - \tan \frac{I}{2}\right)}{\frac{3 \sin 2\theta}{1 - \frac{2}{5} \theta^2} - \frac{I}{2} \sec \frac{I}{2} \left[3 \sin \left(\frac{I}{2} + \theta\right) - \sin \left(\frac{I}{2} - 3\theta\right)\right] + \left[\frac{I}{2} - 3\theta\right)}$$

finally, assume a convenient value for θ and solve for R.

While this is rather a formidable looking equation to work out, it can easily be seen that the 1st and 2nd quantities in the denominator multiplied by R at once give S and A B, so that nearly everything for the subsequent laying out of the curves is found by the one calculation.

In the spring of 1894 the writer in a letter to the "Editor Engineering News" suggested that the above might be an improvement on those given in the pamphlet, and received a reply from the late Mr. A. M. Wellington which is interesting as showing the strong practical grasp he had of all engineering subjects. He states in his letter: "I thought at first you had this simplified into an acceptable form, and you will see intended to publish it, but I fear it is still too complex to make it suitable to lay before engineers as a practicable and practical method. A suggestion! Forget the lemniscata or any other particular curve—Assume a curve in which A S P and H V bisect each other N A P $= \theta$ T P A $= 2\theta$ and V P P′V′ is the original main curve of any length whatever, and you have something that can be called general and proper. Then assume your curve V V′ already run, and H V to be assumed at discretion, and work out all the other elements for laying out the curve either by homologons offsets from tangents and main curve, or by angles from P or A. All this is simple. I have done it and used it, but I have no time to put it in shape. Do it, and you will do a good work. Less won't do—too complex for no gain.

Signed, A. M. WELLINGTON.

The writer has tried to work out a curve using the assumptions suggested by Mr. Wellington, but has not on the whole succeeded in obtaining as simple a set of formulae as may be derived from those numbered (1)′ to (6)′, and which may therefore fairly be called formulae for *modified Lemniscata.*

Looking at equation (1)′ and remembering that 2θ is always a little greater than Sin 2θ and S a little greater than C, within the limits over which θ will likely range, the curve represented by (1)′ will be very slightly altered by writing:

$$2\theta = \frac{s}{3R} \quad \text{or} \quad \theta = \frac{s}{6R} \qquad (a)$$

By equation (3)′ $OH = R + K = \frac{R}{2} (3\cos\theta - \cos 3\theta)$

expanding cos θ and cos 3θ into a series it will be found that

$$K = \frac{3}{2} R (\theta^2 - \theta^4),$$

as K for modified will always be slightly greater than K for true Lemniscota θ^4 can be discarded and therefore

$$\theta^2 = \frac{2K}{3R} \quad \text{or} \quad \theta = \sqrt{\frac{2K}{3R}} \qquad (b)$$

combine (α) and (β) $\therefore K = \frac{s^2}{24 R}$ (c)

combine (a) and (c) $\therefore \theta = \frac{4K}{s}$ (d)

By (5)′ $c = s - \frac{2}{5} s\,\theta^2 = s - \frac{2}{5}\, s\, \frac{16 K^2}{s^2} = s - \frac{32}{5} \frac{K^2}{s}$ (e)

Looking at diagram —$\therefore AH = c \cos \theta - R \sin 3\,\theta$

$$= c\left(1 - \frac{\theta^2}{2}\right) - R\left(3\,\theta - \frac{9}{2}\,\theta^3\right)$$

$$= c\ \left(1 - \frac{8K^2}{s^2}\right] - \frac{s}{2} + 12\,\frac{K^2}{s} \text{ by } (a) \text{ and } (d)$$

$$= c - \frac{8K^2}{s} - \frac{s}{2} + 12\,\frac{K^2}{s^2} \text{ very nearly.}$$

$$= s - \frac{32}{5}\,\frac{K^2}{s} - \frac{8K^2}{s} - \frac{s}{2} + \frac{12\,K^2}{s} \text{ by } (c)$$

$$\therefore AH = \frac{s}{2} - \frac{5}{2}\,\frac{K^2}{s} \quad (f)$$

Equation (a) may be written

$$\frac{\theta}{60{,}5730} = \frac{s}{6\,\frac{5730}{D}}$$

where θ is the number of minutes contained in the angle, and D $= \frac{5730}{R}$

That is θ in minutes $= \frac{S\,D}{10}$ a very important formulae, and the one that would likely generally be used in calculating θ.

The complete set of formulae for modified lemniscata are therefore:

(1) $K = \frac{s^2}{24\,R}$

(2) $\theta = \frac{s}{6\,R} = \frac{4\,K}{s} = \sqrt{\frac{2\,K}{3\,R}}$

(3) θ in minutes $= \frac{s\,D}{10}$ where $D = \frac{5730}{R}$

(4) $AH = \frac{s}{2} - \frac{5}{2}\,\frac{K^2}{s}$

(5) $c = s - \frac{32}{5}\,\frac{K^2}{s}$

(6) deflection to point distant l from $A = \theta \frac{l^2}{s^2} = \theta'$

do do $2l$ do $= 4\theta \frac{l^2}{s^2} = 4\theta'$

do do nl do $n^2 \theta \frac{l^2}{s^2} = n^2 \theta'$

Each of the above equations contains three unknown quantities, any two of which may be assumed at will, so that the curve is extremely flexible and can be made to fulfil almost any conditions.

The correction $\left(\frac{5}{2} \frac{K^2}{s}\right)$ for the distance A H given in equation (4) need only be used when K is fairly large. It is the want of such a correction that has caused trouble (in the case of high degrees of curve) when the otherwise admirable method (first made known to the Society by Mr. Wicksteed) is used.

The accuracy of the above formulae can best be illustrated by an example taken at random.

Let $I = 102° - 0'$

" $R = 410.65$ corresponding to about a 14° curve

" $K = 27$

by (1) $s^2 = 24\ K\ R$

$\therefore 2 \log s = \log 24 + \log K + \log R \quad \therefore s = 515.85$

$\therefore \log s = 2.7125244$

by equation (3) $\theta' = \frac{SD}{10} = \frac{515.85}{10} \frac{5729.58}{410.65}$

$\therefore \log \theta' = \log s + \log 5729.58 - \log 4106.5$

$\therefore \log \theta' = 2.857\ 1733$

$\therefore \quad \theta = 719.736$ minutes $= 11° \ 59' \ 44''$

$$\frac{K^2}{s} = \frac{27^2}{515.85} = 1.413$$

$\therefore$ by (4) $AH = \frac{s}{2} - \frac{5}{2} \frac{K^2}{s} = 257.925 - 3.533 = 254.39$

by (5) $AP = s - \frac{32}{5} \frac{K^2}{s} = 515\ 85 - 9.04 = 506.81$

$A\ B = (R + K) \tan \frac{I}{2} + A\ H$

$= 437.65 \tan 51° + 254.39 = 794.85$

now $AA' = 2\ AB \cos \frac{I}{2} = 2,\ 794.85 \cos 51^\circ = 1000.13$

but AA' should also $= 2\left(AP\left(\cos 51^\circ - \theta\right) + R \sin\left(\frac{I}{2} - 3\theta\right)\right)$

$$= 2\ (506.81 \cos 39^\circ\ 0'\ 16'' + 410.65 \sin 15^\circ\ 0'\ 48'')$$

$$= 2\ (393\ 84 + 106.38)$$

$$= 1000\ 44$$

proving that all the equations given above are almost exactly right, and that if chained correctly the curves would close precisely.

In conclusion the writer would draw attention to a rather peculiar coincidence in connection with the curve represented by the above formulae.

As already mentioned, a short review of the writer's pamphlet was given in "The Railroad Gazette," June 30th, 1893, while in the same journal's issue of August 4th, 1893, there appeared an article entitled: "A New Transition Curve,—The Lemniscata," by Charles H. Tutton, Dept. Public Works, Bureau of Engineering, Buffalo, N.Y.

Mr. Tutton was unaware that anything had been contributed on this curve prior to his paper on the subject, but, hearing in November of the same year that such was not the case, at once wrote, enclosing his own article and asking for the writer's in exchange.

In his letter acknowledging receipt of this, Mr. Tutton gives some very interesting information in connection with his own work in the matter.

He states: "I investigated the curve several years ago, and wrote quite a letter to Prof. Jamieson over two years ago on it, immediately following his series of articles on the Cubic Parabola in the "American Engineer and Railroad Journal."

Its investigation started from Prof. Airy's article in vol. 1 of Van Nostrand's Magazine for 1867, when a set of tables for the Cubic Parabola were given.

The close approximation of the deflections to an arithemical progression at once struck me, and for a time I worked on the idea of developing a curve whose equation could be written $2\ K\ \theta = S^2$ where $K =$ some constant, or in plain English one whose deflection angles from the main tangent were in direct arithmetical progression for equal distances chained along the curve. This curve comes *very close* to the lemniscata, but I have never been able to throw it into what you would term usable shape. Another curve that promised very fair results could be written $2\ K \tan \theta = s^2$ and can be plotted by laying off the tangents of the angles in arithmetical progression for equal arcs.

"I think, altogether, the lemniscata is the preferable curve for field use to any that I know of. I have seen handbooks on transition curves alone that were larger than the entire field book (Henck for instance) usually carried by the engineer, and my experience is that while in the multitude of counsellors there may be safety, yet in the multitude of tables there is a decided chance of error, increasing in rather more than arithmetical progression."

After making some further comparisons Mr. Tutton concludes: "I am busier now on sewers and pavements than on track problems, but if you have leisure I have an idea that the development of the curve $2K\theta = S^2$ might prove interesting, as it is flatter at the beginning and sharper at the centre than the lemniscata."

The coincidence the writer spoke of consists in the fact that the curve represented by the above equation is identical with that dealt with in this paper, and called for want of a better name the "modified Lemniscata." Thus, Mr. Tutton's foresight in the matter has been, in the writer's opinion, amply justified.

That the two curves are one and the same can easily be seen for calling the deflection angle to a point on curve θ, and the deflection angle to the point where transition joins main curve θ' the corresponding distances along curve being S and S′.

by (6) $\theta = \dfrac{\theta' s^2}{s'^2}$

$$\therefore s^2 = \frac{s'^2}{\theta'}\,\theta = 2K\theta \text{ where } 2K = \frac{s'^2}{\theta'}$$

The writer has not touched upon the question of the actual laying out the curves in the field, for the reason that this has been very fully entered into by Mr. Wicksteed and others, whose methods and conclusions apply also to the curve which the writer has attempted to describe.

The only contribution, therefore, that the writer has to offer to the Society is a complete set of properly tabulated general formulae, which will ensure the transition and main curves closing under any and all conditions.